Topology and its Applications 157 (2010) 2542–2552

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topological games and continuity of group operations

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ARTICLE INFO

Article history:
Received 22 February 2010
Received in revised form 1 August 2010
Accepted 1 August 2010

Keywords:
Paratopological group
Topological game
Pseudocompact space
Quasi-continuity

ABSTRACT

We consider a topological game G_Π involving two players α and β and show that, for a paratopological group, the absence of a winning strategy for player β implies the group is a topological one. We provide a large class of topological spaces X for which the absence of a winning strategy for player β is equivalent to the requirement that X is a Baire space. This allows to extend the class of paratopological or semitopological groups for which one can prove that they are, actually, topological groups.
Conditions of the type "existence of a winning strategy for the player α" or "absence of a winning strategy for the player β" are frequently used in mathematics. Though convenient and satisfactory for theoretical considerations, such conditions do not reveal much about the internal structure of the topological space where they hold. We show that the existence of a winning strategy for any of the players in all games of Banach–Mazur type can be expressed in terms of "saturated sieves" of open sets.

1. Introduction

By a space we understand a regular topological T_1-space. We use the terminology from [4,15]. By ω we denote the set $\{0, 1, 2, \ldots\}$. The closure of a subset A in a topological space X will be denoted by $\mathrm{cl}_X A$. If there is no danger of ambiguity, the closure will be denoted simply by $\mathrm{cl}\, A$.

A *paratopological* group is a group endowed with a topology such that the multiplication is jointly continuous. Recall that a *semitopological* group is a group with a topology such that the multiplication is separately continuous.

In 1936 D. Montgomery [19] has proved the following two theorems:

Theorem 1.1. *Every completely metrizable separable semitopological group is a topological group.*

Theorem 1.2. *Every completely metrizable semitopological group is a paratopological group.*

In 1957 R. Ellis [14] established that every locally compact semitopological group is a topological group. Further results on semitopological and paratopological groups were established by Z. Zhelazko [25], N. Brandt [10], H. Pfister [21], L.G. Brown [11], A. Bouziad [7–9], P. Kenderov, I.S. Kortezov, and W.B. Moors [17], E.A. Reznichenko [22] and many other authors (for more information see, for instance, [4,2,3]).

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1 The author was partially supported by Grant DO - 02-360/2008 of the National Fund for Scientific Research of the Bulgarian Ministry of Education, Youth and Science.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.08.001

Various new classes of spaces over which the theorems of D. Montgomery and R. Ellis can be extended were defined either by some traditional topological properties, or by requiring that, in certain topological games, there exists a winning strategy for one of the players (or that there is no winning strategy for the other player) [4,2,3,7,8,16,17,24]. Below we also provide some results of this kind. We introduce a topological game G_Π (played by players α and β) which is closely related to the continuity of the inverse operation in a group. In particular, we show that, if in a paratopological group the player β does not have a winning strategy in the game G_Π, then the group is actually a topological one (Corollary 3.4).

Given a Baire space X and a game G on it, we consider a "relaxed game $G^\sim$" and show (Theorem 4.3) that the existence of a winning strategy for player α in $G^\sim$ implies that there is no winning strategy for player β in G. This allows to describe a very large class of topological spaces for which Montgomery–Ellis-type results are valid (Theorem 4.6). For instance, we show that, if a semitopological group is a Baire space and belongs to this class, then it is a topological group and a paracompact p-space (Theorem 6.9). The class consists of what we call "*spaces with star separation in a compact space*" (Definition 4.4) and contains all p-spaces (see [1]) as well as all "*spaces with countable separation*" (see [18, p. 213]). In particular, all Borel subset of a compact space and all metrizable spaces belong to this class.

When dealing with conditions like 'existence (or non-existence) of winning strategy', one remains with the feeling that they do not completely reveal the intrinsic, purely topological, structure of the spaces in which they operate. We show (Theorem 5.1) that the existence of a winning strategy for player α in a Banach–Mazur-type game G is equivalent to the presence in the space of a "saturated sieve" of open sets with decreasing "sieve sequences" having some additional property determined by the winning rule of the game G. The player β has a winning strategy in the same game G if, and only if, some open subset of the space admits a saturated sieve where none of the sieve sequences has the property related to the winning rule of G (Theorem 5.2).

2. Topological games

Let X be a topological space. Each topological game in X is described by two types of rules: the *playing rules*, that determine how to play the game, and the *winning rule* which determines the winner. In the topological games we consider below, the playing rules coincide with the playing rules of the classical *Banach–Mazur game*. The winning rule differs from game to game and, actually, identifies the game.

Let us first recall the playing rules of the Banach–Mazur game where two players, α and β, select alternatively non-empty open subsets of X. The player α begins the game by selecting the whole space X. Then the player β responds by selecting some non-empty open subset U_0 of X. In turn, player α picks up an open non-empty subset V_0 of U_0. Each time one of the players makes a move (selects an open non-empty set), the other player selects after that an open non-empty subset of the set chosen by the other player. Continuing the game in this way, the two players generate a decreasing sequence $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ of open non-empty subsets of X. This sequence is called *a play* (the set X is the same for all plays we consider and may be omitted, to simplify notation).

Definition 2.1. Player α is said to have won the play $\{(U_n, V_n)\colon n \in \omega\}$ in the *Banach–Mazur game*, if $\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} V_n \neq \emptyset$. Otherwise player β is declared to be the winner in the play.

The Banach–Mazur game will be denoted below by $G_{BM(X)}$. When there is no danger of ambiguity, we will use the notation G_{BM}.

We will need also the notion of a "strategy for a player". This notion does not depend on the winning rule and will have the same meaning in all games we consider.

By *a strategy* s for player α we mean a "rule" that specifies each move of player α in every possible situation. A more precise definition of this notion will be given below in the section for strategies and sieves.

The play $p = \{(U_n, V_n)\colon n \in \omega\}$ is said to be an *s-play*, if the moves of player α were made according to the strategy s. A strategy s for player α is called a *winning strategy in the game G*, if player α wins each s-play according to the winning rule of the game G. If such a strategy s exists, the space X is called *(α, G)-favorable*. We will freely use also the expression "*X is α-favorable for the game G*".

By a *strategy* t for player β we mean a "rule" that specifies each move of player β in every possible situation. More precise definition will be given below, in the section for strategies and sieves.

If the play $p = \{(U_n, V_n)\colon n \in \omega\}$ has been played according to the strategy t of the player β, then it is called a *t-play*. A strategy t for player β is called *a winning strategy in the game G*, if player β wins each t-play. If such a strategy exists, the space X is called *(β, G)-favorable*. The space X is called *(β, G)-unfavorable*, if player β does not have a winning strategy in the game G. This means that, for every strategy t of player β, there is a t-play which is won by α. Evidently, each (α, G)-favorable space is (β, G)-unfavorable. The inverse implication is not valid. There are spaces X for which both players do not have a winning strategy for the Banach–Mazur game G_{BM}.

The next result is known as Banach–Oxtoby theorem. Its proof can be found in [20] (see also [23,12,13,24]).

Theorem 2.2. *A space X is (β, G_{BM})-unfavorable, if and only if it is a Baire space (i.e. the intersection of any countable family of open and dense subsets of X is again dense in X).*

We give now some examples of games we are interested in.

Let $\mathcal{P}$ be any property of decreasing sequences $\{H_n: n \in \omega\}$ of non-empty open subsets of the space X. It determines on X a game $G_{\mathcal{P}}$ with the following winning rule:

Definition 2.3. Player α *wins* the play $p = \{(U_n, V_n): n \in \omega\}$, if the sequence $\{V_n: n \in \omega\}$ has the property $\mathcal{P}$. Otherwise β has won the play.

An immediate example is the Banach–Mazur game G_{BM}.

Definition 2.4. A decreasing sequence $\{H_n: n \in \omega\}$ of open subsets of X has the property *BM* if $\bigcap\{H_n: n \in \omega\} \neq \emptyset$.

Let $\{H_n: n \in \omega\}$ be a decreasing sequence of subsets of X. Put $\operatorname{Lim}\{H_n: n \in \omega\} := \bigcap\{\operatorname{cl} H_n: n \in \omega\}$.

Definition 2.5. A decreasing sequence $\{H_n: n \in \omega\}$ of non-empty open sets in X has the property Π, if $\operatorname{Lim}\{W_n: n \in \omega\} \neq \emptyset$ whenever $\{W_n: n \in \omega\}$ is a decreasing sequence of non-empty open sets such that $W_n \subseteq H_n$ for each $n \in \omega$.

The corresponding game will be denoted by G_Π.

A space X is called *feebly compact*, if every locally finite family of non-empty open subsets in X is finite. For completely regular spaces, feeble compactness is equivalent to pseudocompactness (every continuous function on X is bounded). A normal space is feebly compact if, and only if, it is countably compact. In a feebly compact space every decreasing sequence of non-empty open sets has property Π.

Definition 2.6. Let Y be a dense subspace of a space X. We say that a decreasing sequence $\{H_n: n \in \omega\}$ of open subsets of X has the property S_Y, if each sequence $\{y_n \in Y \cap H_n: n \in \omega\}$ has an accumulation point in X. We write S instead of S_X.

The game corresponding to the property S_Y is denoted by G_{S_Y}, and the game corresponding to property S is denoted by G_S.

In a countably compact space X every sequence of open non-empty sets has the property S.

Definition 2.7. A decreasing sequence $\{H_n: n \in \omega\}$ of open non-empty subsets of a space X has the property C, if $K := \operatorname{Lim}\{H_n: n \in \omega\}$ is a non-empty compact subset of X and, for every open set $U \supset K$, there is some $n \in \omega$ such that $H_m \subset U$ for $m \geqslant n$.

The game corresponding to this property will be denoted by G_C.

In a compact space X, every decreasing sequence of open non-empty sets has the property C.

Each sequence $\{H_n: n \in \omega\}$ with property C has property S and, hence, it has property Π. Therefore, a winning strategy for α-player in G_C is automatically winning in G_S, G_{S_Y} and G_Π. Also, if X is (β, G_C)-unfavorable, then it is (β, G)-unfavorable for $G = G_{S_Y}$, and $G = G_\Pi$.

Note that the properties formulated in Definitions 2.5, 2.6 and 2.7 do not imply, in general, that $\bigcap\{H_n: n \in \omega\} \neq \emptyset$. This will be so, if $\operatorname{cl} H_{n+1} \subset H_n$ for each $n \in \omega$. Since the space X is supposed to be regular, every strategy which is winning for player α or for player β can be modified so that the new strategy is again winning for the same player and produces plays $\{(U_i, V_i): i \in \omega\}$ satisfying the inclusion $\operatorname{cl} V_{n+1} \subset V_n$ for each $n \in \omega$. Therefore, if X is α-favorable for some of the games G_C, G_{S_Y}, G_Π, then it is α-favorable for Banach–Mazur game G_{BM} as well. Similarly, a winning strategy t for player β in the game G_{BM} can be modified to become winning (for the same player) in all the other games considered above. This implies that every $(\beta, G_{\mathcal{P}})$ -unfavorable space, where $\mathcal{P} \in \{\Pi, S_Y, C\}$, is a Baire space as well.

3. Continuity of group operations

Following [17], we call a space X *strongly Baire*, if it is (β, G_{S_Y})-unfavorable for some dense subset $Y \subseteq X$.

The next statement coincides with Theorem 2 from [17].

Theorem 3.1. *Let X be a semitopological group. If X is strongly Baire, then X is a topological group.*

The conclusion of this theorem holds also, if the group X contains some dense strongly Baire subspace (then X itself is a strongly Baire space too).

To prove this theorem one first shows that in every strongly Baire semitopological group the multiplication is jointly continuous, i.e. X is a paratopological group. The second step is to show that the inversion is continuous as well. Both steps of the proof make intensive use of a continuity notion the roots of which could be traced back to V. Volterra (see [6, p. 95]). R. Baire [6] used this notion to show that separately continuous real-valued functions have points of (joint) continuity.

Definition 3.2. A mapping $f : X \to Y$ of a space X into a space Y is called *quasicontinuous at a point* $b \in X$ if for each open neighborhood V of $f(b)$ in Y and every open neighborhood U of b in X there exists a non-empty open set W in X such that $W \subset U$ and $f(W) \subset V$. The mapping f is called *quasicontinuous*, if it is quasicontinuous at each point of X.

In fact, Lemma 3 from [17] states that in a strongly Baire paratopological group G the inversion is quasicontinuous. The continuity of inversion follows then automatically from the observation that a quasicontinuous inversion must be continuous, if the group is paratopological.

The next statement is a generalization of the just mentioned Lemma 3 from [17] and, as we will see later, of Theorem 5.1 from [2].

Theorem 3.3. *Let X be a paratopological group in which the inversion operation is not quasicontinuous. Then the player β has a winning strategy in the game G_Π.*

Proof. If the inversion is not quasicontinuous at the unit element e, then there exists an open neighborhood $V \ni e$ such that $V \cap V^{-1}$ does not contain open subsets. Since X is a regular space and paratopological group there exists some open $U \ni e$ such that $\mathrm{cl}(UU) \subset V$. We have $\mathrm{cl}(U \cap U^{-1}) \subset (U \cap U^{-1})U^{-1} \subset U^{-1}U^{-1} \subset V^{-1}$ and $\mathrm{cl}(U \cap U^{-1}) \subset \mathrm{cl}\, U \subset V$. Thus $\mathrm{cl}(U \cap U^{-1}) \subset V \cap V^{-1}$. This implies that the set $U \cap U^{-1}$ is nowhere dense in X.

Take some open $W \ni e$ such that $\mathrm{cl}(WW) \subset U$ and put $O := W \setminus \mathrm{cl}(U \cap U^{-1})$. Clearly, we have $O \subset W \subset \mathrm{cl}\, O \subset \mathrm{cl}\, W \subset U$. This implies that e does not belong to OU. Indeed, if for some $o \in O$ and $u \in U$ we have $e = ou$, then $U \supset O \ni o = u^{-1} \in U^{-1}$. This contradicts the definition of the set O.

Let us now define a strategy for the player β. As a first choice of β take $U_0 := O$. Suppose the answer of α is an arbitrary non-empty open set $V_0 \subset U_0$. Let $x_0 \in V_0$. Since $e \in \mathrm{cl}\, O$, the open set $V_0 \cap x_0 O$ is not empty. This allows the player β to select some non-empty open set U_1 such that $U_1 \subset \mathrm{cl}\, U_1 \subset V_0 \cap x_0 O$. Proceeding inductively, we can define a strategy for player β which generates plays $\{(U_i, V_i)\colon i \in \omega\}$ and sequences $\{x_i : i \in \omega\}$ satisfying, for every $k \geqslant 0$ the following conditions:

(i) $U_{k+1} \subset \mathrm{cl}\, U_{k+1} \subset V_k \cap x_k O$;
(ii) $x_k \in V_k \subset U_k$.

We show next that every play satisfying the last two conditions is won by β in the game G_Π.

Suppose this is not the case and some play $\{(U_i, V_i)\colon i \in \omega\}$ is won by player α in the game G_Π. Then the set $F := \mathrm{Lim}\{V_n\colon n \in \omega\} = \bigcap\{U_i\colon i \in \omega\} = \bigcap\{\mathrm{cl}\, U_i\colon i \in \omega\}$ is not empty and the sequence $\{U_i\colon i \in \omega\}$ has property Π. Put $A := FW$ and consider the decreasing sequence of open sets $W_k := (G \setminus \mathrm{cl}\, A) \cap U_K$. If all of these sets were non-empty, then (by property Π) we would have that the non-empty set $\Phi := \mathrm{Lim}\{W_n\colon n \in \omega\} \subset (G \setminus A)$. On the other hand, $\Phi \subset F$. This contradiction shows that, for some $k \geqslant 1$, we have $U_k \subset \mathrm{cl}\, A$.

By construction $F \subset U_{k+2} \subset x_{k+1} O \subset x_{k+1} W$. Hence $A = FW \subset x_{k+1} WW$ and, therefore, $\mathrm{cl}\, A \subset x_{k+1}\, \mathrm{cl}(WW) \subset x_{k+1} U$. Since $x_{k+1} \in x_k O$, we get from here that $x_k \in \mathrm{cl}\, A \subset x_{k+1} U \subset x_k OU$. This implies that $e \in OU$ which is a contradiction. The proof is completed. $\square$

Corollary 3.4. *If the paratopological group X is a (β, G_Π)-unfavorable topological space, then the inversion is quasicontinuous and, therefore, X is a topological group.*

Proof. As mentioned above, in a paratopological group, the quasicontinuity of the inversion implies its continuity. $\square$

Remark 3.5. For paratopological groups X which are homeomorphic to a G_δ subset of a pseudocompact space the conclusion of Corollary 3.4 has been proved by Arhangel'skii and Reznichenko in [5] (see also [4]). Under the more general assumption that the underlying space X contains a dense *fan complete* subspace the same statement was proved in [3] (see also [2]). Note that spaces containing dense fan complete subspaces are (α, G_Π)-favorable and, therefore, (β, G_Π)-unfavorable. In the next section we will exhibit a large class of topological spaces which are (β, G_Π)-unfavorable. Among them there are spaces which are unfavorable for both players. For instance, the famous Bernstein set $B \subset [0, 1]$ which has the property that every perfect compact in $[0, 1]$ intersects both B and its complement $[0, 1] \setminus B$, considered as a topological space, is unfavorable for both players in G_Π. A topological subgroup X of the real line sharing the mentioned Bernstein-set property was exhibited in [3]. This topological group is unfavorable for both players too.

4. β-unfavorable spaces

In this section we give a sufficient condition for a space X to be $(\beta, G_{\mathcal{P}})$-unfavorable, and exhibit a large class of spaces for which this condition is fulfilled.

Definition 4.1. Let $\mathcal{P}$ be a property of decreasing sequences of open sets. We say that a decreasing sequence of non-empty open sets $\{H_i: i \in \omega\}$ has the property $\mathcal{P}^{\sim}$, if either $\bigcap\{H_i: i \in \omega\} = \emptyset$ or the sequence has the property $\mathcal{P}$. The corresponding game $G_{\mathcal{P}^{\sim}}$ will be referred to as the "Relaxation of the game $G_{\mathcal{P}}$".

We will see below that, in the relaxed game, player α often has a winning strategy.

Definition 4.2. A property $\mathcal{P}$ of decreasing sequences of non-empty open sets is called *stable*, if whenever $\{W_n: n \in \omega\}$ has property $\mathcal{P}$, then every decreasing sequence $\{V_n: n \in \omega\}$ of open non-empty sets such that $V_n \subset W_n$, for each $n \in \omega$, also has the property $\mathcal{P}$.

The properties Π, S and C are stable. The property BM is not stable. Our interest in stable properties is partially based on the next statement. Stable properties will however play a role in the section for α-favorable spaces too.

Theorem 4.3. *Let $\mathcal{P}$ be a stable property, and X be a Baire space which is $(\alpha, G_{\mathcal{P}^{\sim}})$-favorable. Then X is $(\beta, G_{\mathcal{P}})$-unfavorable.*

Proof. Let $s = \{s_n: n \in \omega\}$ be a winning strategy for α-player in the game $G_{\mathcal{P}^{\sim}}$, and $t = \{t_n: n \in \omega\}$ be an arbitrary strategy for player β. We will prove that t is not a winning strategy. This will be done by considering a new strategy $t' = \{t'_n: n \in \omega\}$ for player β which is obtained by "composing" the strategies t and s. We define this strategy inductively.

Let U_0 be the first move of β under t. Denote by W_0 the response of α under s. We let W_0 be the first move of β under the new strategy t'. Let V_n, $n \geqslant 0$, be a move of player α following a t'-move of β at the n-th stage of the game. Player β applies t and selects some open $U_{n+1} \subset V_n$. In response, player α applies s to obtain an open set W_{n+1} which will be considered to be the choice of β under the new strategy t'. In this way each t'-play $\{(W_i, V_i): i \in \omega\}$ is accompanied by a sequence of sets $\{U_i: i \in \omega\}$ such that:

a) $\{(U_i, W_i): i \in \omega\}$ is an s-play;
b) $\{(U_i, V_i): i \in \omega\}$ is a t-play.

Since X is a Baire space, there exists some t'-play $\{(W_i, V_i): i \in \omega\}$ for which $\bigcap\{W_i: i \in \omega\} = \bigcap\{V_i: i \in \omega\} \neq \emptyset$. Thus, the s-play $\{(U_i, W_i): i \in \omega\}$ is won by α, and the sequence $\{W_i: i \in \omega\}$ has the property $\mathcal{P}$. Since $\mathcal{P}$ is a stable property, and $V_i \subset W_i$ for every $i \in \omega$, it follows that the sequence $\{V_i: i \in \omega\}$ has the property $\mathcal{P}$. Hence, the t-play $\{(U_i, V_i): i \in \omega\}$ is won by α. □

We will use now this result to describe a very large class of spaces which are β-unfavorable for the game $G_{\mathcal{P}}$ where $\mathcal{P}$ is any of the properties $\{C, S, \Pi\}$ introduced above.

Given a point x and a family δ of open subsets of some space, then $\mathrm{St}(x, \delta) = \bigcup\{W \in \delta: x \in W\}$. This set is *the star of x with respect to δ*.

Definition 4.4. Let $X \subset Z$. The space X is said to have *star separation in Z*, if there exist families $\{\delta_n: n \in \omega\}$ of open subsets of Z which separate points of X from points of $Z \setminus X$ in the following sense: for every pair of points $x \in X$ and $z \in Z \setminus X$, there exists $n \in \omega$ such that at least one of the stars $\mathrm{St}(x, \delta_n)$, $\mathrm{St}(z, \delta_n)$ is not empty and contains only one of the two points.

If the families $\{\delta_n: n \in \omega\}$ form a star separation for X in Z, then they form a star separation for $Z \setminus X$ in Z as well. Every open subset U of any space Z has an evident star separation in it, the set U itself. Hence, every closed subset of Z also has star separation in Z. It is easy to see that the collection of all sets with star separation in a certain space Z is closed under taking countable unions and countable intersections. Moreover, Souslin scheme, applied to sets with star separation in Z, also produces a set with a star separation in Z.

The spaces admitting a star separation in some compact space Z by families δ_n which are covers of X have already been studied. A space with such star separation is called *a p-space* [1] (or also *feathered space*). The class of p-spaces is very large. For instance, all metric spaces are p-spaces.

Spaces X admitting star separation in a compact space Z by families δ_n each consisting of just one open subset of Z have been used in the study of fragmentability and σ-fragmentability of Banach spaces under the name *spaces with countable separation* (see [18, p. 213]).

Theorem 4.5.

a) *Let X be a dense subset of a compact space Z and let X have a star separation in Z. Then the space X is $(\alpha, G_{C^{\sim}})$-favorable.*
b) *Let X be a dense subset of a countably compact space Z and let X have a star separation in Z. Then the space X is $(\alpha, G_{S^{\sim}})$-favorable.*
c) *Let X be a dense subset of some feebly compact space Z (in particular, of some completely regular pseudocompact space Z), and let X have a star separation in Z. Then the space X is $(\alpha, G_{\Pi^{\sim}})$-favorable.*

Proof. Let the families $\{\delta_n\colon n \in \omega\}$ form a star separation for the space X in Z and let X be dense in Z. We define inductively a strategy $s = \{s_n\colon n \in \omega\}$ for player α which will turn out to be a winning one for the corresponding game in a), b) and c).

Let the first move of β be an arbitrary non-empty open set U_0 of X. Take some open $U_0' \subset Z$ such that $\mathrm{cl}_Z U_0' \cap X \subset U_0$ and either $U_0' \cap \{\bigcup W\colon W \in \delta_0\} = \emptyset$ or $\mathrm{cl}_Z U_0' \subset W$ for some set W from the family δ_0. Such a set U_0' exists because X is dense in Z. Put $s_0(U_0) := V_0 := U_0' \cap X$.

Proceeding inductively, we construct the strategy $s = \{s_n\colon n \in \omega\}$ so that every s-play $\{(U_n, V_n)\colon n \in \omega\}$ in X is accompanied by a sequence $\{U_n'\colon n \in \omega\}$ of open subsets of Z such that, for every $k \in \omega$, we have:

i) $\mathrm{cl}_Z U_k' \cap X \subset U_k$;
ii) Either $U_k' \cap \{\bigcup W\colon W \in \delta_k\} = \emptyset$ or there is some $W \in \delta_k$ for which $\mathrm{cl}_Z U_k' \subset W$;
iii) $V_k = U_k' \cap X$;
iv) $\mathrm{cl}_Z U_{k+1}' \subset U_k'$.

Let $p = \{(U_i, V_i)\colon i \in \omega\}$ be an s-play in X. If $\bigcap\{V_i\colon i \in \omega\} = \emptyset$, then there is nothing to prove. Suppose that $\bigcap\{V_i\colon i \in \omega\} \neq \emptyset$, and let x be a point from this set. We will show that the non-empty set $K := \bigcap\{\mathrm{cl}_Z U_i';\ i \in \omega\}$ is a subset of X. Indeed, for every $n \in \omega$ either $K \cap \{\bigcup W\colon W \in \delta_n\} = \emptyset$, or there is $W \in \delta_n$ which contains K. In the first case, the stars of the points from K with respect to δ_n are empty and cannot separate the points of K. In the second case, the stars of all points from K contain the set W and also do not separate the points of K. Hence, either $K \subset Z \setminus X$ or $K \subset X$. Since $x \in K \cap X$, we conclude that $K \subset X$. Then $K = \bigcap\{V_i\colon i \in \omega\}$. If Z is a compact space, then K is compact and the sequence $\{V_n\colon n \in \omega\}$ has the property C. This finishes the proof of case a). Evidently, if Z is countably compact, then the sequence $\{V_n\colon n \in \omega\}$ has the property S. This settles case b). To consider case c), suppose Z is feebly compact and $\{W_n\colon n \in \omega\}$ is a decreasing sequence of non-empty open sets such that $W_n \subseteq V_n$ for each $n \in \omega$. Find, for each $n \in \omega$, some open subset W_n' of Z such that $W_n' \cap X = W_n$ and $W_n' \subset U_n'$. Since Z is feebly compact the set $\mathrm{Lim}\{W_n'\colon n \in \omega\} = \bigcap\{\mathrm{cl}_Z W_n'\colon n \in \omega\}$ is not empty. By construction the latter set is a subset of $K \subset X$. This implies that $\mathrm{Lim}\{W_n\colon n \in \omega\} = \bigcap\{\mathrm{cl}_X W_n\colon n \in \omega\}$ is not empty and, therefore, the sequence $\{V_n\colon n \in \omega\}$ has property Π. The proof of case c) is completed. □

Corollary 4.6.

a) *If X is a dense Baire subspace of a compact space Z, and X has a star separation in it, then X is β-unfavorable for the game G_C.*
b) *If X is a dense Baire subspace of a countably compact space Z, and X has a star separation in it, then X is β-unfavorable for the game G_S.*
c) *If X is a dense Baire subspace of a feebly compact space Z (in particular, of a completely regular pseudocompact space Z), and X has a star separation in it, then X is β-unfavorable for the game G_Π.*

Remark 4.7. The real line equipped with Sorgenfrey topology is a paratopological group, but it is not a topological group. It is also a Baire space. Hence, this space does not have a star separation in any feebly compact space.

Remark 4.8. It is easy to see that a space X has star separation in some Z if, and only if, X has star separation in $\mathrm{cl}_Z X$. In the case when Z is a compact space somewhat more can be proved.

Theorem 4.9. *The completely regular space X has star separation in some compactification bX if, and only if, it has star separation in its Stone–Ĉech compactification βX.*

Proof. Let $f : \beta X \to bX$ be the uniquely determined continuous mapping of βX onto bX such that $f^{-1}(x) = x$ for any point $x \in X$. It is trivial to check that the preimages under f of open sets from some star separation of X in bX form a star separation of X in βX.

Let us now assume that the families of open sets $\{\delta_n\colon n \in \omega\}$ form a star separation for X in βX. For any non-empty open subset U of βX put $U' := bX \setminus f(\beta X \setminus U)$. Note that the open set U' is not empty. It contains the non-empty set $f(U \cap X)$. Also, a point $y \in bX$ belongs to U' if, and only if, $f^{-1}(y) \subset U$.

For each $n \in \omega$ we put $\delta_n' = \{U'\colon U \in \delta_n\}$. Further, let M be the family of all non-empty finite subsets of ω. For any $\mu \in M$ we put $W_\mu = \bigcup\{U\colon U \in \bigcup\{\delta_i\colon i \in \mu\}\}$ and denote by δ_μ' the one-set family $\{W_\mu'\}$. We will show now that, taken together, the families $\{\delta_n'\colon n \in \omega\}$ and $\{\delta_\mu' : \mu \in M\}$ form a star separation for X in bX. Let $x \in X$ and $y \in bX \setminus X$. Let z be an arbitrary point of $f^{-1}(y)$. There exists some $k = k(x,z) \in \omega$ such that at least one of the stars $\mathrm{St}(x, \delta_k)$ and $\mathrm{St}(z, \delta_k)$ is not empty and contains only one of the points x, z. Suppose the set $\mathrm{St}(x, \delta_k)$ is non-empty. Then it does not contain z (otherwise $\mathrm{St}(z, \delta_k)$ would contain x and we would not have separation). This implies that the set $\mathrm{St}(x, \delta_k')$ is non-empty, does not contain y and we have the separation. Therefore, without loss of generality, we may assume that $\mathrm{St}(x, \delta_{k(x,z)}) = \emptyset$ for every $z \in f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there is a finite set $\mu \in M$ such that $f^{-1}(y) \subset \bigcup\{\bigcup\{U\colon U \in \delta_k\}\colon k \in \mu\} = W_\mu$ and $\mathrm{St}(x, \delta_i) = \emptyset$ for any $i \in \mu$. Clearly, W_μ' contains y and does not contain x. Hence the families $\{\delta_n', \delta_\mu'\colon n \in \omega,\ \mu \in M\}$ form a star separation for X in the compactification bX of the space X. □

We give here an example of a space X which has star separation (even countable separation) in every compact space but fails to be a p-space. It is the well-known "Michael Line".

Example 4.10. Let R be the real line with its usual topology. Consider in R a stronger topology obtained by adding to the usual topology all irrational numbers as isolated points. R with this new topology will be denoted by X. On the set of rational numbers $Q \subset R$ both topologies induce one and the same topology.

Let bX be any compactification of X. The set $X \setminus Q$ of irrational numbers is open in bX. For every pair r', r'' of different rational numbers denote by $U(r', r'')$ some open subset of bX such that $U(r', r'') \cap R$ is the open interval having r' and r'' as end-points. Let $x \in X$ and $y \in bX \setminus X$. If x is irrational, then the set $R \setminus Q$ contains x but not y. If x is rational then there exists some $U(r', r'')$ containing x but not y. This means that the countable collection of open sets $X \setminus Q$, $\{U(r', r''): r', r'' \in Q\}$ separates the points of X from the points of $bX \setminus X$ (i.e. X has countable separation in bX).

Suppose that X admits star separation in bX by families $\{\delta_n: n \in \omega\}$ open (in bX) sets so that each δ_n is a cover for X. For each $n \in \omega$ and $r \in Q$ fix some open interval $I_n(r)$ in R such that $r \in I_n(r) \subset \mathrm{cl}_{bX} I_n(r) \subset U$ for some open U, $U \in \delta_n$. Note that $I_n(r) \subset \mathrm{cl}_{bX} I_n(r) \subset U \subset \mathrm{St}(x, \delta_n)$ for every real number $x \in I_n(r)$. Consider the set $V_n = \bigcup\{I_n(r): r \in Q\}$. It is open and dense in the usual topology of R. By Baire category theorem there exists some irrational number $x \in \bigcap\{V_n: n \in \omega\}$. Then $x \in \bigcap\{I_n: n \in \omega\}$ where $I_n = I_n(r)$ for some $n \in \omega$ and $r \in Q$. Find a sequence $\{x_i: i \in \omega\}$ of different irrational numbers which converges to x in the usual topology of R and $A_n := \{x_i: i \geqslant n\} \subset I_n$ for each $n \in \omega$. Clearly, $\mathrm{cl}_{bX} A_n \subset \mathrm{cl}_{bX} I_n \subset \mathrm{St}(x, \delta_n)$. We have, therefore, $\emptyset \neq \bigcap\{\mathrm{cl}_{bX} A_n: n \in \omega\} \subset \bigcap\{\mathrm{St}(x, \delta_n): n \in \omega\}$. On the other hand, each A_n is a closed subset of X and, hence, $\mathrm{cl}_{bX} A_n \cap X = A_n$. It follows that $\bigcap\{\mathrm{cl}_{bX} A_n: n \in \omega\} \cap X = \emptyset$. Therefore the point $x \in X$ cannot be separated from the points of $\bigcap\{\mathrm{cl}_{bX} A_n: n \in \omega\} \subset bX \setminus X$. This shows that X is not a p-space.

We conclude this section by a proposition which shows that a space admitting star separation in a compact space is, in a certain sense, close to paracompact p-space.

Theorem 4.11. *Let a space X have a star separation in some compact space. Then there exists a (possibly empty) G_δ-subspace Y of X which is a paracompact p-space and $X \setminus Y$ is a union of countably many closed nowhere dense subspaces of X.*

The proof of this proposition will be given after Corollary 6.8 in the section about α-favorable spaces.

5. Strategies and sieves

In this section we show that the existence of a strategy for the player α (the existence of a strategy for player β) is equivalent to the presence in the topological space (in some open subset of the space) of a structure known under the name "sieve".

Under a *sieve* in a space X we understand a sequence $\gamma = \{\gamma_n: n \in \omega \cup \{-1\}\}$ of families $\gamma_n = \{V_\mu: \mu \in A_n\}$ of non-empty open subsets of X, and a sequence of mappings $\pi = \{\pi_n : A_n \to A_{n-1}: n \in \omega\}$ such that:

a) the index set A_{-1} is a singleton and the family γ_{-1} contains only one set, the whole space X;
b) $V_\mu \subset V_{\pi_n(\mu)}$, for every $\mu \in A_n$ and $n \in \omega$.

We call the sieve (γ, π) *saturated*, if the following condition holds:

c) for every $n \in \omega$ and every non-empty open set $U \subset V_\mu$, where $\mu \in A_{n-1}$, there exists some $\mu' \in A_n$ such that $\pi_n(\mu') = \mu$ and $V_{\mu'} \subset U$.

Note that, for a saturated sieve (γ, π) and for any $n \in \omega \cup \{-1\}$, the set $\bigcup\{V_\mu: \mu \in A_n\}$ is dense in X.

Any sequence $\{V_{\mu_n}: \mu_n \in A_n,\ n \in \omega \cup \{-1\}\}$ for which $\mu_{n-1} = \pi_n(\mu_n)$, for every $n \in \omega$, will be called a *coordinated sieve sequence* or a *cs-sequence*.

We also need a precise definition of the notion of a *strategy for player α*. Under a strategy s for player α we understand a sequence of mappings $s = \{s_n: n \in \omega\}$. Each s_n determines the n-th move of player α. The domain $\mathrm{Dom}\, s_0$ of s_0 consists of all non-empty open sets U_0 in X. The value $s_0(U_0)$ of s_0 at U_0 is the response of α to the choice U_0 by player β. The inclusion $s_0(U_0) \subset U_0$ should be satisfied. The domain $\mathrm{Dom}\, s_1$ of s_1 consists of all pairs U_0, U_1 where $U_0 \in \mathrm{Dom}\, s_0$ and U_1 is an arbitrary non-empty open subset of $s_0(U_0)$. U_1 is a possible second move of β, if his/her first move was U_0. The answer of α is the set $s_1(U_0, U_1)$ which is a non-empty open subset of U_1. In general, the domain $\mathrm{Dom}\, s_n$ and the values of the mapping s_n satisfy, for $n \geqslant 2$, the following conditions:

d) $\mathrm{Dom}\, s_n$ consists of all finite sequences of non-empty open sets $(U_0, \ldots, U_{n-1}, U_n)$ such that $(U_0, \ldots, U_{n-1}) \in \mathrm{Dom}\, s_{n-1}$ and $U_n \subset s_{n-1}(U_0, \ldots, U_{n-1})$;
e) $s_n(U_0, \ldots, U_{n-1}, U_n) \subset U_n$.

Theorem 5.1. *Let X be a topological space.*

i) *Every strategy for player α generates a saturated sieve in the space X. Conversely, any saturated sieve in X determines a strategy for player α;*
ii) *The space X is $(\alpha, G_{\mathcal{P}})$-favorable if, and only if, there exists a saturated sieve such that every cs-sequence has the property $\mathcal{P}$.*

Proof. Given a strategy $s = \{s_n\colon n \in \omega\}$ for player α, we take A_{-1} to be an arbitrary singleton, and γ_{-1} to be the family consisting of just one set, the space X. Further we put, for $n \in \omega$, $A_n := \operatorname{Dom} s_n$, $\gamma_n = \{V_\mu = s_n(\mu)\colon \mu \in A_n\}$, and determine the mappings $\pi_n : A_n \to A_{n-1}$, $n \geqslant 1$, by the relation

$$\pi_n(U_0, \ldots, U_{n-1}, U_n) = (U_0, \ldots, U_{n-1}).$$

The mapping π_0 sends all elements of the set $A_0 = \operatorname{Dom} s_0$ into the singleton A_{-1}.

Evidently, (γ, π) where $\gamma = \{\gamma_n\colon n \in \omega \cup \{-1\}\}$ and $\pi = \{\pi_n\colon n \in \omega\}$ is a saturated sieve. For every s-play $\{(U_n, V_n)\colon n \in \omega\}$ and for every $n \in \omega$ we have $V_n = V_\mu$ for some $\mu = (U_0, \ldots, U_n) \in A_n$. This establishes a one-to-one correspondence between all s-plays and all *cs*-sequences. Moreover, the s-play is won by α in the game $G_{\mathcal{P}}$ if, and only if, the corresponding *cs*-sequence has the property $\mathcal{P}$. Hence, strategy s is winning in game $G_{\mathcal{P}}$ precisely when all *cs*-sequences have the property $\mathcal{P}$.

Let a saturated sieve $(\gamma = \{\gamma_n\colon n \in \omega \cup \{-1\}\}, \pi = \{\pi_n\colon n \in \omega\})$ be given in X. We will define inductively a strategy for player α. The first move of α is X. The first move of β is some non-empty open set $U_0 \subset X = V_\mu$, $\mu = A_{-1}$. Since the sieve (γ, π) is saturated, there exists some $\mu' \in A_0$ such that $\pi_0(\mu') = \mu$ and $V_{\mu'} \subset U_0$. Select one such μ' and put $s_0(U_0) := V_{\mu'}$. Let U_1 be an arbitrary non-empty subset of $s_0(U_0) = V_{\mu'}$. Since the sieve is saturated, there exists $\mu'' \in A_1$ such that $\pi_1(\mu'') = \mu'$ and $V_{\mu''} \subset U_1$. Choose such μ'' and put $s_1(U_0, U_1) := V_{\mu''}$. Proceeding inductively, we determine a strategy $s = \{s_n\colon n \in \omega\}$ for player α in such a way that, for every s-play $p = \{(U_i, V_i)\colon i \in \omega\}$, the sequence $\{V_i\colon i \in \omega\}$ is a *cs*-sequence. The latter sequence will have the property $\mathcal{P}$ if, and only if, the play p is won by player α in the game $G_{\mathcal{P}}$. $\square$

A statement similar to Theorem 5.1 is valid also for strategies of player β. Any strategy t for player β is a sequence of mappings $\{t_n\colon n \in \omega\}$ having non-empty open subsets of X as values. Each mapping t_n determines the n-th move of player β. The domains and values of these mappings satisfy the following requirements. The domain $\operatorname{Dom} t_0$ of t_0 consists of all possible first moves of player α. I.e. $\operatorname{Dom} t_0$ is $\{X\}$. The only value $t_0(X)$ of the mapping t_0 is some open non-empty set $U_0 \subset X$. The domain $\operatorname{Dom} t_1$ of t_1 consists of all pairs (X, V_0) where V_0 is an arbitrary non-empty open subset of $U_0 = t_0(X)$. The values of t_1 satisfy the condition $t_1(X, V_0) \subset V_0$ for every $(X, V_0) \in \operatorname{Dom} t_1$. In general, $\operatorname{Dom} t_{n+1}$ for $n \geqslant 1$ consists of all finite sequences $(X, V_0, V_1, \ldots, V_{n-1}, V_n)$ where $(X, V_0, V_1, \ldots, V_{n-1}) \in \operatorname{Dom} t_n$ and V_n is an open non-empty subset of $t_{n-1}(X, V_0, V_1, \ldots, V_{n-1})$. The values of t_n satisfy the condition $t_n(X, V_0, V_1, \ldots, V_{n-1}, V_n) \subset V_n$.

Theorem 5.2.

i) *Every strategy for player β generates a saturated sieve in some non-empty open subset U of X. Conversely, any saturated sieve in a non-empty open set $U \subset X$ determines a strategy for player β;*
ii) *The space X is $(\beta, G_{\mathcal{P}})$-favorable if, and only if, there exists an open non-empty set $U \subset X$ and a saturated sieve in it such that no cs-sequence has the property $\mathcal{P}$.*

Proof. Given a strategy $t = \{t_n\colon n \in \omega\}$ of player β, we observe that, after the first choice of β, all the choices of the two players that follow are open subsets of the set $U_0 = t_0(X)$. Consider U_0 as a space and denote the Banach–Mazur game on it by $BM(U_0)$. The strategy t determines a strategy $s = \{s_n\colon n \in \omega\}$ for player α in the game $BM(U_0)$ by putting $s_n = t_{n+1}$ for every $n \in \omega$. According to Theorem 5.1, the strategy s determines a saturated sieve in U_0 with *cs*-sequences corresponding precisely to t-plays in Banach–Mazur game in X. A *cs*-sequence of this sieve in U_0 has the property $\mathcal{P}$ precisely when the corresponding t-play is won by player α.

Conversely, consider a saturated sieve in some non-empty open set $U \subset X$. According to Theorem 5.1, to this sieve corresponds a certain strategy $s = \{s_n\colon n \in \omega\}$ for player α in the Banach–Mazur game $BM(U)$. Define a strategy for player β in $BM(X)$ by putting $t_0(X) = U_0 := U$ and $t_{n+1} = s_n$ for $n \in \omega$. To each t-play there corresponds a *cs*-sequence. If the latter sequence does not have property $\mathcal{P}$, then the t-play is won by β. $\square$

Corollary 5.3. *Let X be a space and $\mathcal{P}$ a property of decreasing sequences of open sets. The space X is $(\beta, G_{\mathcal{P}})$-unfavorable if, and only if, for every non-empty open set $W \subset X$ and every saturated sieve in W there exists some cs-sequence with the property $\mathcal{P}$.*

6. α-favorable spaces

The major instruments in this section are some sieves (γ, π) for which $\gamma_n = \{V_\mu\colon \mu \in A_n\}$, $n \in \omega$, are disjoint families of open sets. Though not necessarily saturated, such sieves also can determine strategies for player α and allow to explore more closely the topological properties of the underlying space.

Given a strategy s for player α, we denote by $P(s)$ the set of all s-plays.

Definition 6.1. We call a subset A of $P(s)$ *concurrent*, if for every pair of plays $p = \{(U_n, V_n)\colon n \in \omega\}$, $p' = \{(U'_n, V'_n)\colon n \in \omega\}$ from A the following condition holds:

$(*)$ if for some integer $k \in \omega$ we have $V_k \cap V'_k \neq \emptyset$, then $U_j = U'_j$ and $V_j = V'_j$ for every j, $0 \leqslant j \leqslant k$. I.e. the two plays coincide (or "run together") up to the k-th coordinate.

Singletons (the one-play sets) are the simplest examples of concurrent subsets of $P(s)$. Clearly, the usual inclusion provides an inductive order in the family of all concurrent subsets of $P(s)$. Denote by $A(s)$ some maximal concurrent subset of $P(s)$, and put, for $n \in \omega$,

$$\gamma_n := \left\{V \subset X\colon V = V_n \text{ for some play } p = \left\{(U_i, V_i)\colon i \in \omega\right\} \in A(s)\right\}.$$

The next statement summarizes the properties of the collections $\{\gamma_k\colon k \in \omega\}$. Each of these properties follows immediately from the definition of γ_k, the maximality of the concurrent set $A(s)$, or can be established by induction.

Proposition 6.2.

a) *For each $n \in \omega$, the family γ_n consists of disjoint non-empty open sets*;
b) *Every $V \in \gamma_{n+1}$ is contained in precisely one set from γ_n*;
c) *For every sequence $\{V_i\colon i \in \omega\}$ such that $V_{i+1} \subset V_i$ and $V_i \in \gamma_i$ for every $i \in \omega$, there exists exactly one play $p \in A(s)$ such that $p = \{(U_i, V_i)\colon i \in \omega\}$*;
d) *If, in addition, the strategy s is winning for player α in the game $G_{\mathcal{P}}$, then each sequence $\{V_i\colon i \in \omega\}$ such that $V_{i+1} \subset V_i$ and $V_i \in \gamma_i$ for every $i \in \omega$ has the property $\mathcal{P}$*;
e) *For each $n \in \omega$, the set $L_n = \bigcup\{V\colon V \in \gamma_n\}$ is dense in X.*

Property b) from Proposition 6.2 determines a mapping $\pi_n : \gamma_{n+1} \to \gamma_n$, $n \in \omega$, that puts into correspondence to each $V \in \gamma_{n+1}$ the only set $V' \in \gamma_n$ for which $V \subset V'$. Clearly, (γ, π), where $\gamma = \{\gamma_n\colon n \in \omega\}$, and $\pi = \{\pi_n\colon n \in \omega\}$ form a sieve in the space X. Because of the properties listed in Proposition 6.2, we call such sieves *Dense Disjoint sieves* or, simply, *DD-sieves*.

Remark 6.3. As we have mentioned earlier, every strategy $t = \{t_n\colon n \in \omega\}$ for player β in a game played on a space X generates a strategy for player α in a game played in the open set $U_0 = t_0(X)$ (the initial move of β under the strategy t). Hence, t generates a DD-sieve in U_0.

Theorem 6.4. *Let $\mathcal{P}$ be a stable property of decreasing sequences of open sets. Then the following conditions are equivalent*:

(i) *X admits a winning strategy for player α in the game $G_{\mathcal{P}}$.*
(ii) *There exists a DD -sieve $(\gamma = \{\gamma_n\colon n \in \omega\}, \pi = \{\pi_n\colon n \in \omega\})$ on X such that every cs-sequence $\{V_i \in \gamma_i\colon i \in \omega\}$ in X has property $\mathcal{P}$.*

Proof. Let s be a winning strategy for player α in the game $G_{\mathcal{P}}$. The DD-sieve defined in Proposition 6.2 satisfies condition (ii).

It remains to show that (ii) implies (i). Let $n \in \omega$ and $U_n \subset X$ be the n-th move of β. Because of Proposition 6.2 e), there exists some $V_n \in \gamma_n$ such that $U_n \cap V_n \neq \emptyset$. Select one such V_n and let the answer of α be the set $U_n \cap V_n$. This strategy generates plays which, due to the stability of property $\mathcal{P}$, are won by player α. $\square$

Remark 6.5. The strategy of player α defined in the proof of Theorem 6.4 has a special property. The choices of α depend only on the choice U_n of player β at n-th move. The previous moves of the two players do not matter. Strategies depending only on the set U selected by β and on the number n of the move are called *Markov strategies*. Thus, each of the conditions (i) and (ii) in Theorem 6.4 is equivalent to the next condition:

(iii) There exists a winning Markov strategy for the player α in the game $G_{\mathcal{P}}$.

In the rest of the section, we show that there is a relationship between metric spaces and some $(\alpha, G_{\mathcal{P}})$-favorable spaces. We will illustrate this connection in the case when $\mathcal{P}$ is one of the properties C, $C^{\sim}$, S, and Π. To see this, we endow the set $P(s)$ of all plays generated by a given strategy s with the well-known Baire metric which was already used for similar purposes in [18]. It is defined as follows:

For $p = \{(U_i, V_i)\colon i \in \omega\}$ and $p' = \{(U'_i, V'_i)\colon i \in \omega\}$ we put $d(p, p') = 0$ if $p = p'$ and otherwise $d(p, p') = 1/(n+1)$, where $n = \min\{k\colon U_k \neq U'_k\}$.

Note that, if $d(p, p') < 1/(n+1)$, then $U_i = U'_i$ for all i such that $0 \leqslant i \leqslant n$. This implies that also $V_i = V'_i$ for all i such that $0 \leqslant i \leqslant n$, because the sets V_i are determined uniquely by the strategy s.

It is easy to see that, with respect to this metric, the space $P(s)$ is a complete metric space. Note also that the open set $\{p \in P(s)\colon d(p_0, p) < n^{-1}\} = \{p \in P(s)\colon d(p_0, p) \leqslant (n+1)^{-1}\}$ is also closed.

Theorem 6.6. *Let $\mathcal{P}$ be any of the properties C, S. If the space X admits a winning strategy s for player α in the game $G_{\mathcal{P}}$, then there exist a dense G_δ subset $L \subset X$, a complete metric space Z, and a mapping $f : L \to Z$ such that f is continuous, closed and onto.*

If $\mathcal{P} = C$ ($\mathcal{P} = S$), then for every $z \in Z$ the preimage $f^{-1}(z) := \{p\colon f(p) = z\}$ is a compact (countably compact) space.

Proof. Since X is regular we may assume, without loss of generality, that for each s-play $p = \{(U_i, V_i)\colon i \in \omega\}$ we have $\operatorname{cl} V_{i+1} \subset V_i$, for every $i \in \omega$. Moreover, since $\mathcal{P}$ is either C or S, we have that for an s-play also the following properties hold:

(i) $\bigcap\{V_i\colon i \in \omega\}$ is not empty;
(ii) For every open $U \supset \bigcap\{V_i\colon i \in \omega\}$ there exists some $n \in \omega$ such that $\bigcap\{V_i\colon i \in \omega\} \subset V_n \subset U$ (i.e. $\{V_i\colon i \in \omega\}$ is a base of neighborhoods of the set $\bigcap\{V_i\colon i \in \omega\}$).

Let $A(s)$ be some maximal concurrent subset of $P(s)$. It is easy to see that $A(s)$ is a closed subset of $(P(s), d)$. Therefore, the space $Z := (A(s), d)$ is a complete metric space.

Using the notation from Proposition 6.2, we put $L_n := \bigcup\{V\colon V \in \gamma_n\}$ for $n \in \omega$, and denote by L the set $\bigcap\{L_n\colon n \in \omega\}$. Since X is (α, BM)-favorable, it is a Baire space. Clearly, L is a dense G_δ subset of X.

Every $x \in L$ and $n \in \omega$ uniquely determine some $V_n \in \gamma_n$ such that $x \in V_n$. This, in turn, identifies uniquely an s-play $f(x) = \{(U_i, V_i)\colon i \in \omega\}$ which belongs to the maximal concurrent set $A(s)$ (see Proposition 6.2). It is easy to see that the mapping $f : L \to Z$ so defined is continuous. The strategy s produces plays with non-empty intersection, since s is winning. Therefore, f is onto. It is also closed. This follows from requirement (ii) above.

Evidently, the fibers $f^{-1}(z)$ are compact (countably compact), if $\mathcal{P} = C$ ($\mathcal{P} = S$, respectively). □

A mapping f which is continuous, closed and with compact fibers is called *perfect*.

Theorem 6.7. *Suppose the Baire space X admits a winning strategy s for player α in the game $G_{C^\sim}$. Then there exist a dense G_δ subset $L \subset X$, a metric space Z, and a perfect mapping f of L into Z. In particular, X contains dense G_δ subset which is a paracompact p-space.*

Proof. Reasoning as in the proof of the previous theorem (and using the same notations) we find the open and dense sets $\{L_n\colon n \in \omega\}$. Since X is Baire the set $L = \bigcap\{L_n\colon n \in \omega\}$ is dense in X. The mapping f maps L into Z and has compact fibers. As preimage of a metric space under a perfect mapping the subset L is a paracompact p-space. □

Corollary 6.8. *Suppose the Baire space X has star separation in a compact space. Then X contains a dense G_δ subset Y which is a paracompact p-space.*

Proof. Follows immediately from Theorem 4.5 a) and the previous theorem. □

We are now ready to give a proof of Theorem 4.11.

Proof of Theorem 4.11. Let $\delta = \{\delta_n\colon n \in \omega\}$ be a star separation of X in some compact space bX. Consider a maximal disjoint family γ of open first Baire category subsets U of X. The set $E = \bigcup\{U\colon U \in \gamma\}$ is an open first Baire category subset of X. If the interior H of the closed set $X \setminus E$ is empty, there is nothing to prove. X would be the union of countably many closed nowhere dense subsets. Therefore we may assume that H is non-empty. Let H' be an open subset of bX such that $H = H' \cap X$. The collection δ complemented with the set H' makes a star separation for H in bX. Note that H is a Baire space. By the previous corollary H contains a dense G_δ subset Y which is a paracompact p-space. □

Theorem 6.9. *Let X be a Baire semitopological group which has star separation in some compact space. Then X is a topological group and a paracompact p-space.*

Proof. Corollary 4.6 implies that X is strongly Baire. From Theorem 2 of [17] it follows that X is a topological group. By Theorem 6.7 X contains compacts K of the type $f^{-1}(z)$ where f is a perfect mapping into metric space. In this case X itself admits a perfect mapping onto a metrizable space. □

For spaces X which are (α, G_Π)-favorable, there is a result similar to Theorem 6.6. However, the corresponding mapping f has a property which lies between closeness and openness.

Definition 6.10. A set-valued mapping $F : Z \to X$ is said to be upper semi continuous (θ upper semi continuous) at the point $z_0 \in Z$, if for every open set $U \supset F(z_0)$ there is an open neighborhood $W \ni z_0$ such that $\bigcup\{F(z)\colon z \in W\} \subset U$ ($\bigcup\{F(z)\colon z \in W\} \subset \operatorname{cl} U$). We say that F is usc (θ-usc), if it has the corresponding property at every $z \in Z$.

Evidently, upper semi-continuity implies θ upper semi-continuity. If X is a normal space and F is closed-valued, then the two notions coincide.

Note that a mapping $f : L \to Z$ is closed if and only if the inverse mapping $F := f^{-1}$ is usc. Recall also that a subset M of a space X is called *bounded* if for every locally finite family γ of open subsets in X the set $\{U \in \gamma\colon U \cap M \neq \emptyset\}$ is finite.

Theorem 6.11. *If a space X admits a winning strategy s for player α in the game G_Π, then there exist a dense G_δ subset $L \subset X$, a complete metric space Z and a mapping $f : L \to Z$ such that*

a) *f is continuous and onto*;
b) *the inverse mapping $F := f^{-1}$ is θ usc*;
c) *for every $z \in Z$, the preimage $f^{-1}(z) := \{p\colon f(p) = z\}$ is a bounded subset of X.*

Proof. The proof is almost identical with the proof of Theorem 6.6. Only b) and c) need to be proved. To prove b) we take some open $U \subset X$ containing $f^{-1}(p) = \bigcap\{V_i\colon i \in \omega\}$ where $p = \{(U_i, V_i)\colon i \in \omega\}$ is an s-play. It suffices to show that there is some $n \in \omega$ such that $V_n \subset \operatorname{cl} U$. Suppose this is not the case. Then the sets $W_i := V_i \setminus \operatorname{cl} U$, $i \in \omega$, are open and non-empty. By property Π, the set $T = \bigcap\{\operatorname{cl} W_i\colon i \in \omega\}$ is not empty and is contained in $\bigcap\{V_i\colon i \in \omega\} = f^{-1}(p)$. On the other hand, we have $T \subset X \setminus U$, a contradiction. This proves b). To prove c) take an infinite family $\gamma = \{H_i\colon i \in \omega\}$ of open sets intersecting $f^{-1}(p) = \bigcap\{V_i\colon i \in \omega\}$ where $p = \{(U_i, V_i)\colon i \in \omega\}$ is an s-play. It suffices to show that γ is not locally finite. Without loss of generality we can assume that $H_i \subset V_i$ for every $i \in \omega$. For every $i \in \omega$ put $W_i := \bigcup\{H_k\colon k \geqslant i\}$. The set $\operatorname{Lim}\{W_i\colon i \in \omega\}$ is not empty and every open set intersecting it intersect infinitely many members of the family γ. $\square$

Theorem 6.12. *Let X be an (α, G_Π)-favorable space which admits a winning strategy s such that, for each s-play $\{(U_n, V_n)\colon n \in \omega\}$, the set $\bigcap\{V_n\colon n \in \omega\}$ is a singleton. Then there exist a dense G_δ subset $L \subset X$ which is metrizable by a complete metric.*

Proof. The mapping $f : L \to Z$ defined in the proof of Theorem 6.6 is a homeomorphism in this case. $\square$

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